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TWO-DIMENSIONAL ANALYSIS OF PRESSURE TRANSIENTS IN PIPELINES

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SUMMARY

The paper presents a two-dimensional model for the investigation of pressure transients in pipelines. The governing equations have been established and a method of solving the equations using the centre implicit method is presented. The theoretically predicted values are compared with the experimentally determined pressure transients for horizontal pipelines with a valve at the end. The two-dimensional model gives results which are accurate than those of the one-dimensional model and are in good agreement with the experimental results.

KEY WORDS Computational methods Pipelines Pressure transients Two-dimensional model

INTRODUCTION

In the study of pressure transients in pipelines, the one-dimensional model is usually used. The velocity and pressure are assumed to be uniform across the pipe sections and the analysis leads to results which are in agreement with the experimental results at high Reynolds number; however, at low Reynolds number, i.e., $Re < 2000$, it has been found that there is a large discrepancy between the experimental and theoretical results.[1] To eliminate this discrepancy for an abrupt transient, such as a sudden valve closure, Zielke[2] used the concept of a heuristic weighting function to evaluate the wall shear. The instantaneous wall shear stress is determined as the sum of the steady state value and a term in which certain weights are given to the past velocity changes at the pipe cross-section. This correction is applied to the one-dimensional water hammer equations and solved using the method of characteristics.

Another approach has been presented by Ohmi *et al.*[3] where the pressure variation of a slightly compressible fluid in a pipeline is numerically computed using the method of characteristics and a finite difference technique, in which the wall shear is evaluated from the cross-sectional profile of instantaneous axial velocities. They have shown that the distribution of total viscosity in a pulsating pipe flow can be modelled best by a four-region model. Accurate solution of the pressure transients depends on the correct modelling of the total viscosity. In the case of axisymmetric pipe flow the variation of the velocity profile and hence the wall shear stress affects the pressure transients. Therefore, to further improve the accuracy of prediction of pressure transients for laminar flow at low Reynolds number, it is necessary to extend the presently used one-dimensional model[4] to a two-dimensional approach.

This paper presents a two-dimensional model to predict the pressure transients in a pipeline assuming axisymmetric flow. The governing equations are solved by a finite difference technique which includes an artificial viscosity term (a damping factor) presented by the authors[4] and a

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modified viscosity model presented by Ohmi and Usui[5] to account for pipe friction. The theoretical results obtained with the above two-dimensional centre implicit method (CIM) model are compared with the experimental results obtained by other investigators for the case of sudden closure of a valve at the end of a pipeline. The advantages of the CIM model over the method of characteristics have been discussed previously.[4]

THEORETICAL MODEL

To investigate the pressure transient in a pipeline, where the flow is two-dimensional, a simple pipeline system is considered in which a pressure wave is generated by sudden closure of a valve at the end of the pipeline. The pipeline is of uniform and circular cross-section and connected to a constant head reservoir maintained at a predetermined value. It is assumed that the fluid flow is homogeneous and axisymmetrical. The other assumptions are:

(1) The axial velocity component u and the density ρ are functions of the axial x and radial r distances and time t only.
(2) The radial velocity component v is very small compared with the axial component u.
(3) The pressure p in a cylindrical element at any cross-section along the pipeline corresponds to the pressure at the axis of symmetry; therefore it is assumed that the $\partial p/\partial r$ term is negligible and hence $p = p(x, t)$.

Continuity equation

The continuity equation in cylindrical co-ordinates (Figure 1) for axisymmetric flow at radius r is

$$\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial r}+\frac{v}{r}\right)+u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial r}+\frac{\partial \rho}{\partial t}=0. \quad (1)$$

Taking the time derivative of the density and pressure functions, and using the relationship between density, pressure and wave speed c for compressible flow, i.e., $\partial \rho/\partial p = 1/c^2$, and combining with equation (1), the continuity equation reduces to

$$\rho c^2\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial r}+\frac{v}{r}\right)+u\frac{\partial p}{\partial x}+\frac{\partial p}{\partial t}=0. \quad (2)$$

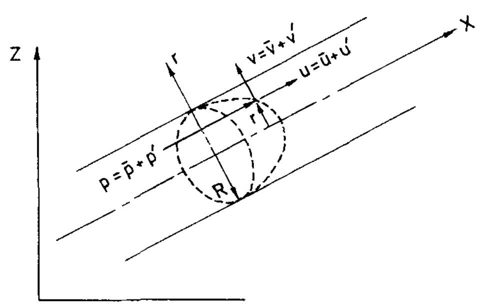

Figure 1. Co-ordinate system for the pipe flow

Momentum equation

For axisymmetric flow the Navier–Stokes equation in the x direction is

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial r}=-\frac{1}{p}\frac{\partial p}{\partial u}-g\frac{\partial z}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)$$

$$+\frac{\nu}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial r}+\frac{v}{r}\right), \tag{3}$$

where g is the acceleration due to gravity, z is the elevation of pipe at any x and ν is the kinematic viscosity.

Since it is assumed that $u \gg v$, the momentum equation for axisymmetric flow in the r direction need not be considered.

Governing equations

The transient flow in a pipeline is unsteady. Therefore the instantaneous values of p, u and v in equations (2) and (3) may be considered as sums of a short-time average value and a fluctuating component:

$$u=\bar{u}+u', \qquad v=\bar{v}+v', \qquad p=\bar{p}+p', \tag{4}$$

where the overbar denotes the short-time average value and the prime denotes the fluctuating component.

Substituting equation (4) into equations (2) and (3), and using a simplification proposed by Brown[6] and applying the assumption $u \gg v$, the two-dimensional governing equations reduce to the form

$$\frac{\partial \bar{p}}{\partial t}+\bar{u}\frac{\partial \bar{p}}{\partial x}+\rho c^2\frac{\partial \bar{u}}{\partial x}=0, \tag{5}$$

$$\frac{\partial \bar{u}}{\partial t}+\bar{u}\frac{\partial \bar{u}}{\partial x}=-\frac{1}{p}\frac{\partial \bar{p}}{\partial x}-g\frac{\partial z}{\partial x}+\nu\left(\frac{\partial^2 \bar{u}}{\partial r^2}+\frac{1}{r}\frac{\partial \bar{u}}{\partial r}\right)-\frac{1}{r}\frac{\partial}{\partial r}(r\overline{u'v'}), \tag{6}$$

where $\overline{u'v'}$ is the Reynolds stress.

The Reynolds stress $\overline{u'v'}=\varepsilon\,\partial u/\partial r$, where ε is the eddy viscosity, when substituted into equation (6) modifies it to

$$\frac{\partial \bar{u}}{\partial t}+\bar{u}\frac{\partial \bar{u}}{\partial x}=-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x}-g\frac{\partial z}{\partial x}+\left(\frac{\partial \nu'}{\partial r}+\frac{\nu'}{r}\right)\frac{\partial \bar{u}}{\partial r}+\nu'\frac{\partial^2 \bar{u}}{\partial r^2}, \tag{7}$$

where $\nu'=\nu+\varepsilon$.

The accurate solution of equation (7) depends on the correct modelling of the distribution of ν' across the pipe. Ohmi and Usui[5] have shown that the distribution of ν' for flow in a pipe can be represented by a four-region model across the cross-section of the pipe.

Distribution model of eddy viscosity

The four-region model describes the eddy viscosity distribution by dividing the cross-section of pipeline into four regions. The eddy viscosity in each region is defined as

$$\frac{v'}{v}=k\frac{u^*Y}{v}+m=kY^*+m, \tag{8}$$

where k, Y^* and m are given below:

Region	Y^*	k	m
I	0–5	0	1
II	5–30	0·2	0
III	30–0·175 R^*	0·4	0
IV	0·175*–R^*	0	0·07R^*

where

$$R^*=u^*R/v, \qquad u^*=\sqrt{(\tau/\rho)}, \qquad \tau=-\rho v'(\partial u/\partial r)_R, \qquad Y^*=u^*Y/v, \qquad Y=R-r.$$

Ohmi and Usui[5] have shown that the above four-region model agrees well with the eddy viscosity model of Taylor, Prandtl and Von Karman. They have also shown that the computed results with the proposed four-region model agree well with the experimental data for pipe flow.

Integration of two-dimensional equations

The two-dimensional governing equations for continuity (equation (5)) and momentum (equation (7)) include three independent variables x, r and t. If the two equations are integrated over the cross-section, defining the wall shear stress $\tau=-(v'\rho(\partial\bar{u}/\partial r))_R$, then the mean pressure P and the mean velocity U reduce to the one-dimensional equations as follows:

$$\frac{\partial P}{\partial t}+U\frac{\partial P}{\partial x}+\rho c^2\frac{\partial U}{\partial x}=0, \tag{9}$$

$$\frac{\partial U}{\partial t}+\frac{1}{\rho}\frac{\partial P}{\partial x}+U\frac{\partial U}{\partial x}+\frac{2}{R\rho}\tau+g\frac{\partial z}{\partial x}=0. \tag{10}$$

The mean pressure and the mean velocity at a cross-section can be calculated from the above equations and the velocity profile from the momentum equation (7).

Finite difference equations

The method of solving equations (9) and (10) using the centre implicit finite difference technique has been outlined by the authors.[4] With this method the mean pressure P and the mean velocity U at any cross-section of a pipe can be determined during a pressure transient; Figure 2(a) shows the notation used in the time–spatial plane. The velocity profile at any cross-section of a pipe can be determined from equation (7) using an appropriate numerical method. In this study the centre implicit method is selected; Figures 2(b) and 2(c) show the notation and the grid used in the radial–axial plane. The derivatives using Taylor's first approximations are

$$\frac{\partial\bar{u}}{\partial t}=\frac{\bar{u}_{i,r}^{t+\Delta t}-\bar{u}_{i,r}^{t}+\bar{u}_{i+1,r}^{t+\Delta t}-\bar{u}_{i+1,r}^{t}}{2\,\Delta t}, \tag{11}$$

$$\frac{\partial\bar{u}}{\partial x}=\frac{(2-\theta)(\bar{u}_{i,r}^{t+\Delta t}-\bar{u}_{i,r}^{t})+\theta(\bar{u}_{i+1,r}^{t+\Delta t}-\bar{u}_{i+1,r}^{t})}{2\,\Delta x}, \tag{12}$$

$$\frac{\partial\bar{u}}{\partial t}=\frac{\bar{u}_{i,r+1}^{t+\Delta t}-\bar{u}_{i,r-1}^{t+\Delta t}}{2\,\Delta t}, \tag{13}$$

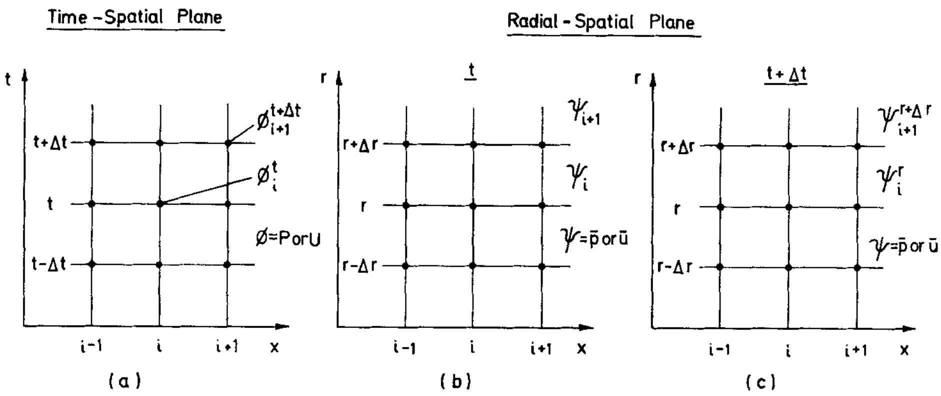

Figure 2. Notation of two-dimensional grid system for the centre implicit method

$$\frac{\partial^2 \bar{u}}{\partial r^2} = \frac{\bar{u}_{i,r+1}^{t+\Delta t} + \bar{u}_{i,r-1}^{t+\Delta t} - 2\bar{u}_{i,r}^{t+\Delta t}}{(\Delta r)^2}, \tag{14}$$

$$\bar{u} = (\bar{u}_{i,r}^{t} + \bar{u}_{i+1,r}^{t})/2. \tag{15}$$

In equation (12) an artificial viscosity term θ is introduced as a damping factor; the significance of this term has been evaluated and explained previously by the authors.[4] If equations (11)–(15) are substituted into equation (7), the finite difference equation becomes

$$B_1 \bar{u}_{i,r+1}^{t+\Delta t} + B_2 \bar{u}_{i,r}^{t+\Delta t} + B_3 \bar{u}_{i,r-1}^{t+\Delta t} = B_4, \tag{16}$$

where

$$B_1 = \left(\frac{\partial v'}{\partial r} + \frac{v'}{r}\right)\frac{1}{2\,\Delta r} + \frac{v'}{(\Delta r)^2}, \tag{16a}$$

$$B_2 = \frac{-2v'}{\Delta r} - \frac{2-\theta}{4\,\Delta x}(\bar{u}_{i,r}^{t} + \bar{u}_{i+1,r}^{t}) - \frac{1}{2\,\Delta t}, \tag{16b}$$

$$B_3 = -\left(\frac{\partial v'}{\partial r} + \frac{v'}{r}\right)\frac{1}{2\,\Delta r} + \frac{v'}{(\Delta r)^2}, \tag{16c}$$

$$B_4 = \frac{\bar{u}_{i+1,r}^{t+\Delta t} - \bar{u}_{i,r}^{t} - \bar{u}_{i+1,r}^{t}}{2\Delta t} + \frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} + g\frac{\partial z}{\partial x}$$

$$+ \frac{\bar{u}[\theta(\bar{u}_{i+1,r}^{t+\Delta t} - \bar{u}_{i+1,r}^{t}) - (2-\theta)\bar{u}_{i,r}]}{2\,\Delta x}. \tag{16d}$$

If the pipe cross-section is divided into M nodes in the r direction, then equation (16) can be applied to nodes 2 to $M-1$. The other two equations are derived using the boundary conditions given below:

$$\text{at node } M,\ r = R: \bar{u} = 0, \quad \text{i.e.,} \quad \bar{u}_{i,M}^{t+\Delta t} = 0; \tag{17a}$$

$$\text{at node } 1,\ r = 0: \frac{\partial \bar{u}}{\partial r} = 0, \quad \text{i.e.,} \quad \bar{u}_{i,1}^{t+\Delta t} = \bar{u}_{i,0}^{t+\Delta t}. \tag{17b}$$

The axial velocity $\bar{u}$ at any section i of the pipe can be determined at time step $t + \Delta t$ by applying equations (16) and (17). The eddy viscosity terms $\partial v'/\partial r$ and v' are computed using the four-region

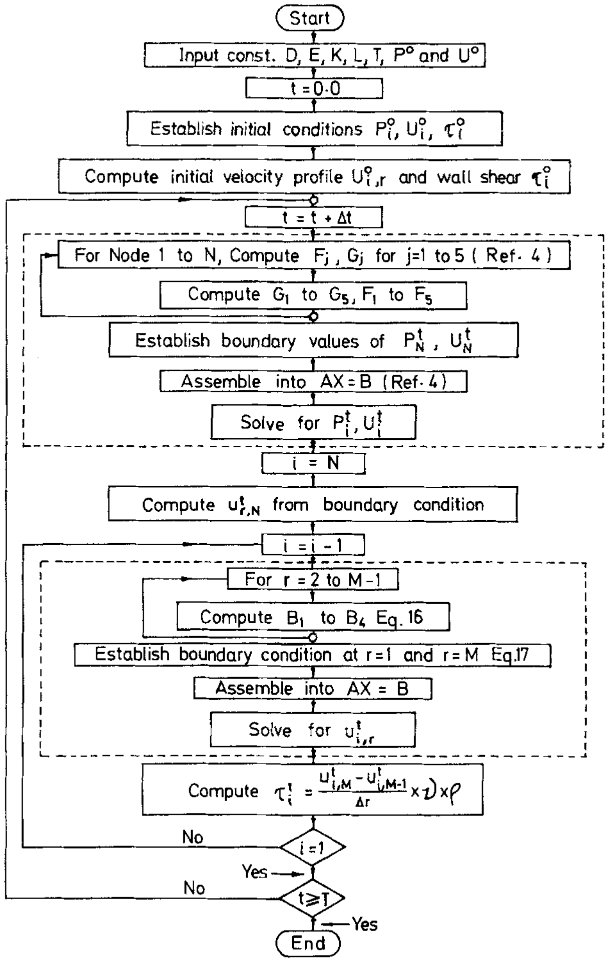

Figure 3. Flow chart for computation procedure

distribution model given in equation (8). In the above equations the axial velocity at section $i+1$, i.e., $\bar{u}_{i+1,r}^{t+\Delta t}$, is required for determining the velocity profile at section i in the x direction. Therefore the velocity profile at section $i+1$ should be computed before the computation of the velocity profile at section i.

A computer program was written in Fortan IV to solve the two-dimensional governing equations of pressure transients in pipelines using the centre implicit method (CIM). The computer program was run on a mainframe computer IBM 3081G at the National University of Singapore. The flow chart, Figure 3, shows the solution procedure used in the computation. The CIM solution procedure and the conditions for solving equations (9) and (10), which represent the one-dimensional analysis of pressure transients, have been presented previously by the authors.[4] The summary of the solution procedure for the two-dimensional governing equations is as follows:

(1) Initial conditions are assumed to be that of the steady state.
(2) Using equations (9) and (10), compute the mean pressure and mean velocity at the next time step, where the wall shear of the last time step is used.
(3) Compute the velocity profile by using equations (16) and (17) for $i = 1$ to $i = N$.

(4) Compute the wall shear from the gradient of the velocity profile just computed at $r = R$.
(5) Based on the computed values of P, U and τ compute the solution for the next time step; and then the procedure is repeated.

EXPERIMENTAL DATA AND COMPUTATIONAL PARAMETERS

To evaluate the above numerical technique for determination of pressure transients at different Reynolds numbers, the experimental results presented by Streeter and Lai[1] and Holmboe and Rouleau[7] are considered. In both cases the experimental results have been obtained for horizontal pipes of uniform cross-section with a valve at the end of the pipeline. The valve is closed instantaneously to create the pressure transient in the pipeline. In addition the pressure transient in a vertical pipeline is experimentally determined in the laboratory; details of this are given by Tan.[8] The schematic layout of the experimental test rig is shown in Figure 4. For the single-phase liquid pressure transient tests the valve between the air reservoir and the air–water injector remains closed.

In the theoretical computation of pressure transients in pipelines, the values established for stability, damping ratio, etc. by the authors are used. The properties of fluid and pipe used by different researchers in their experimental investigation are given in Table I. For all three cases the number of nodes in the x direction (31), the number of nodes in the r direction (20), the stability criterion ($\Delta t = \Delta x/c$) and the damping ratio ($\theta = 1{\cdot}005$) are the same. Other parameters for the three different cases are as follows:

	Case 1 (Ref. 1)	Case 2 (Ref. 7)	Case 3 (Ref. 8)
Spatial increment ratio x/L	0·1	0·1	0·033
Radial increment ratio r/D	0·026	0·026	0·026

DISCUSSION OF RESULTS

The one-dimensional analysis[4] of pressure transients agrees well with the experimental results at high Reynolds number. But at low Reynolds number, where the wall shear stress is significant,

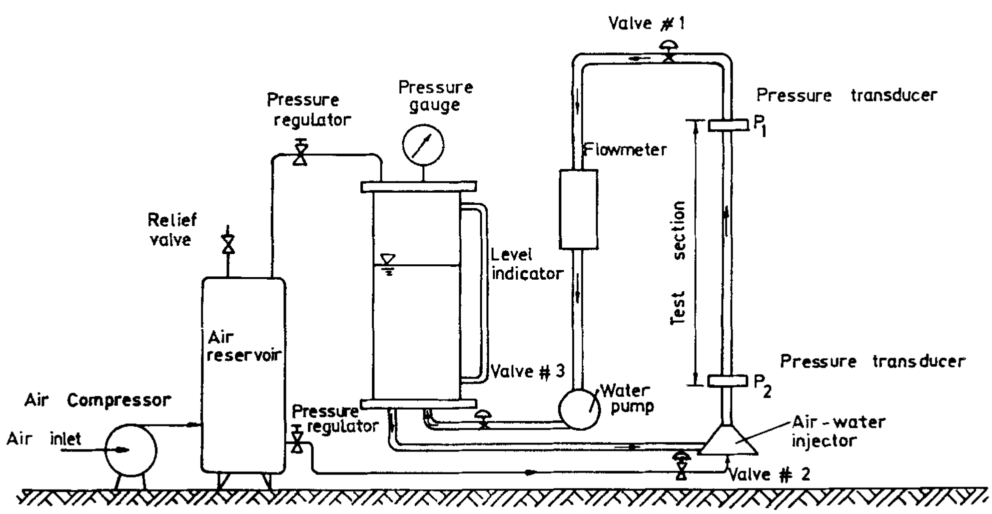

Figure 4. Schematic diagram of the experimental test facility

Table 1. Fluid properties and pipe configuration data

	Case 1 (Ref. 1)	Case 2 (Ref. 7)	Case 3 (Ref. 8)
Fluid	Water	Oil	Water
Viscosity (10^{-6} m^2 s^{-1})	0·959	39·75	1·007
Density (kg m^{-3})	998·9	864·1	1000
Reservoir head (m)	13·72	—	9·58
Pipe length (m)	91·44	36·09	2·0
Pipe diameter (mm)	11·07	25·4	47·5
Mean velocity (m s^{-1})	0·112	0·128	0·0065
Reynolds number	1340	82	307
Pipe material	Copper	Copper	Perspex
Wave speed (m s^{-1})	1318·5	1324·3	594·5

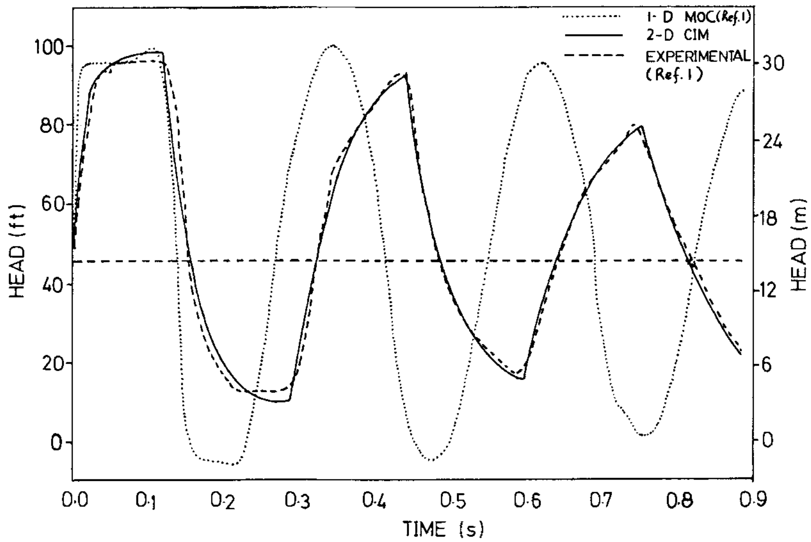

Figure 5. Comparison of experimental[1] and theoretical results at $Re = 1340$

there is a phase shift between the theoretical and experimental results. It is proposed that pressure transients at low Reynolds number should be approached using a two-dimensional CIM model. To verify the proposed method for pressure transients in pipe flow, the experimental data published by other investigators are used in the first instance. Figure 5 shows a comparison between the experimental results[1] and the theoretical results obtained with the method of characteristics and the proposed two-dimensional model for the pressure transient at the valve due to sudden closure. The good agreement between the experimental results and the proposed two-dimensional model shows that the proposed model represents the actual flow characteristics and the wall shear stress better. The centre implicit method is more easily adaptable for two-dimensional flow than is the method of characteristics; furthermore, a computer program can be readily written when the CIM is used.

Figure 6 shows the theoretical results obtained with the proposed model and the experimental results[7] at the centre of a pipeline embedded in concrete; again the pressure transient is created by the sudden closure of a valve. In the same figure the theoretical results obtained using the centre

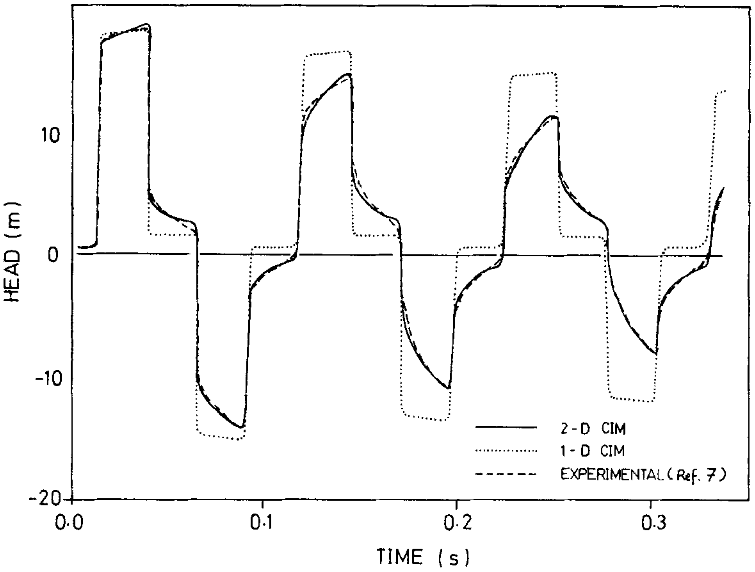

Figure 6. Comparison of experimental[7] and theoretical results at $Re = 82$

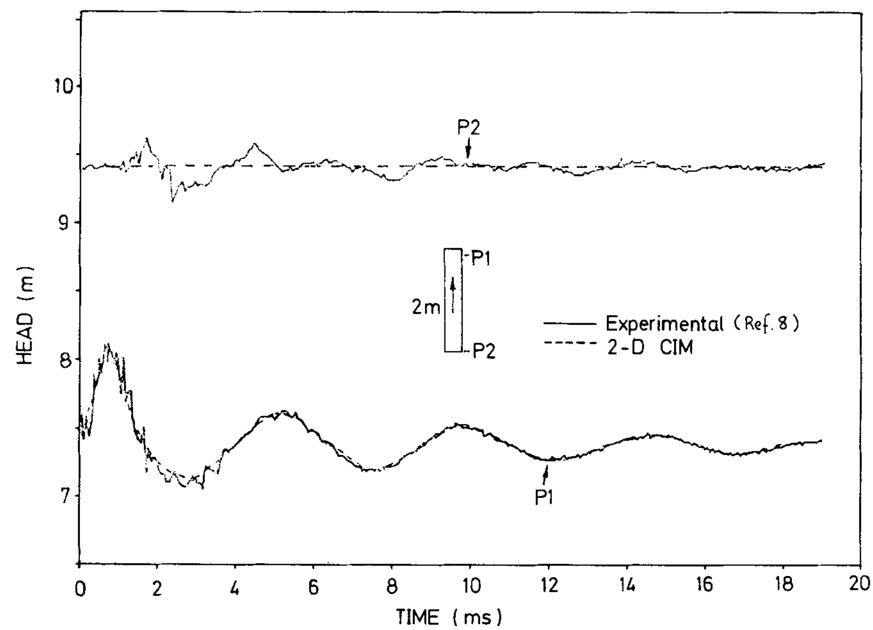

Figure 7. Comparison of experimental[8] and theoretical results at $Re = 307$

implicit method for the one-dimensional model are superimposed to illustrate the discrepancy between the one-dimensional and two-dimensional models. The proposed two-dimensional model gives theoretical values which agree very closely with the experimental results. The discrepancy between the experimental results and the theoretical results obtained using both the one-dimensional centre implicit method and the method of characteristics at low Reynolds number has been highlighted.

Another comparison is made between the theoretical results of the two-dimensional model and the experimental results of the authors determined at low wave speed and with pipe of a different material. In the configuration a piezoelectric pressure transducer was used at each end of a 2 m length vertical Perspex pipe and the pressure transient was created by closing the solenoid valve at the downstream side of the upper transducer. The short length of test section was justifiable owing to the fact that high-speed data acquisition equipment was used. The head in the reservoir was maintained by supplying compressed air. Figure 7 shows the experimental and theoretical results at the upper pressure transducer (P1) but only the experimental results at the lower pressure transducer (P2). In the theoretical calculation the pressure at the lower transducer was assumed to be constant, but this pressure varies slightly as seen in the figure and this has introduced pressure fluctuations at the upper transducer too. Apart from the second-order fluctuations, the amplitude and frequency of the theoretical and experimental values agree well when the pressure transient occurs as a result of the sudden closure of a valve.

CONCLUSIONS

The prediction of pressure transients using the proposed two-dimensional model and the use of the CIM for solving give results which agree well with the experimental results. The proposed method is applicable for axisymmetric flow and over a range of low Reynold numbers and a range of wave speed. Although the computation is performed implicitly, it is necessary to use the stability criterion $\Delta t = \Delta x/c$ and the value of damping ratio $\theta = 1{\cdot}005$ which have been established earlier by the authors.[4,8]

The two-dimensional CIM has been shown to give accurate predictions for pressure transients caused by sudden closure of a valve in a simple pipe configuration. The practical importance of this method can be enhanced if the method is extended to a pipe network type of configuration.

LIST OF SYMBOLS

$B_1, B_2, \ldots$	coefficients
c	celerity, acoustic velocity, sonic velocity
$F_1, F_2, \ldots$	coefficients
$G_1, G_2, \ldots$	coefficients
g	acceleration due to gravity
H	pressure head in height of water column
i	any node in the n direction
k	coefficient in eddy viscosity model
L	length of the pipe
M	number of grids in the r direction
m	coefficient in eddy viscosity model
N	number of nodes in the x direction
P	mean pressure at any cross-section
$\bar{p}$	short-time average of p
p'	fluctuating component of p
R	radius of pipe
r	radial co-ordinate
t	time

u	axial velocity at r
$\bar{u}$	short-time average of u
u'	fluctuating component of u
U	mean velocity at any cross-section
u^*	friction velocity $= \sqrt{(\tau/\rho)}$
v	radial velocity at r
$\bar{v}$	short-time average of v
v'	fluctuating component of v
x	longitudinal co-ordinate (+ ve to the initial flow direction)
Y	$= R - r$
z	elevation
Δr	radial increment
Δt	time increment
Δx	linear distance increment
ε	eddy viscosity
θ	artificial viscosity
μ	viscosity
ν	kinematic viscosity
ν'	total viscosity $= \varepsilon + \nu$
ρ	density of the liquid
τ	wall shear

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